Elementary Algebra: Solving Linear Equations in One Variable

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Abstract

Elementary Algebra: An introduction to solving linear equations in one variable.

1 Module Overview

Learning how to solve various algebraic equations is one of our main goals in algebra. This module introduces the basic techniques for solving linear equations in one variable. *(Prerequisites: Working knowledge of real numbers and their operations.)*

Objectives

- **Define Linear Equations in One Variable**
- **Solutions to Linear Equations**
- **Solving Linear Equations**
- **Combining Like Terms and Simplifying**
- **Literal Equations**

2 Define Linear Equations in One Variable

We begin by establishing some definitions.

Definition 1: Equation
An equation is a statement indicating that two algebraic expressions are equal.

Definition 2: Linear Equation in One Variable
A linear equation in one variable x is an equation that can be written in the form $\text{ax} + b = 0$ where a and b are real numbers and $a \neq 0$.

Following are some examples of linear equations in one variable, all of which will be solved in the course of this module.

$$x + 3 = -5 \quad (1)$$

$$\frac{x}{3} + \frac{1}{2} = \frac{2}{3} \quad (2)$$

$$5\,(3x+2)-2=-2\,(1-7x) \tag{3}$$

3 Solutions to Linear Equations in One Variable

The variable in the linear equation $2x+3=13$ is x. Values that can replace the variable to make a true statement compose the solution set. Linear equations have at most one solution. After some thought, you might deduce that $x=5$ is a solution to $2x+3=13$. To verify this we substitute the value 5 in for x and see that we get a true statement, $2\,(\mathbf{5})+3=10+3=13$.

Example 1
Is $x=3$ a solution to $-2x-3=-9$?
Yes, because $-2\,(3)-3=-6-3=-9$

Example 2
Is $a=-\frac{1}{2}$ a solution to $-10a+5=25$?
No, because $-10\left(-\frac{1}{2}\right)+5=5+5=10\neq 25$

When evaluating expressions, it is a good practice to replace all variables with parenthesis first, then substitute in the appropriate values. By making use of parenthesis we could avoid some common errors using the order of operations.

Example 3
Is $y=-3$ a solution to $2y-5=-y-14$?

$$\begin{array}{rclc} 2\,(\ \ )-5 & = & -(\ \ )-14 & \text{Replace variables with parenthesis.} \\ 2\,(-3)-5 & = & -(-3)-14 & \text{Substitute the appropriate value.} \\ -6-5 & = & 3-14 & \text{Simplify.} \\ -11 & = & -11 & \text{True.} \end{array}$$

Yes because $y=-3$ produces a true mathematical statement.

4 Solving Linear Equations in One Variable

When the coefficients of linear equations are numbers other than nice easy integers, guessing at solutions becomes an unreasonable prospect. We begin to develop an algebraic technique for solving by first looking at the properties of equality.

4.1 Properties of Equality

Given algebraic expressions A and B where c is a real number:

Property 1: Addition Property of Equality

$$\text{If } A=B \text{ then } A+c=B+c$$

Property 2: Subtraction Property of Equality

$$\text{If } A=B \text{ then } A-c=B-c$$

Property 3: Multiplication Property of Equality

$$\text{If } A = B \text{ and } c \neq 0 \text{ then c}A = cB$$

Property 4: Division Property of Equality

$$\text{If } A = B \text{ and } c \neq 0 \text{ then } \frac{A}{c} = \frac{B}{c}$$

NOTE: Multiplying or dividing both sides of an equation by zero is carefully avoided. Dividing by zero is undefined and multiplying both sides by zero will result in an equation 0=0.

To summarize, the equality is retained if we add, subtract, multiply and divide both sides of an equation by any nonzero real number. The central technique for solving linear equations involves applying these properties in order to isolate the variable on one side of the equation.

Example 4

Use the properties of equality to solve: $x + 3 = -5$

$$\begin{array}{ccll} x+3 & = & -5 & \\ x+3-3 & = & -5-3 & \text{Subtract 3 on both sides.} \\ x & = & -8 & \text{Simplify} \end{array}$$

The solution set is $\{-8\}$.

Example 5

Use the properties of equality to solve: $-5x = -35$

$$\begin{array}{ccll} -5x & = & -35 & \\ \frac{-5x}{-5} & = & \frac{-35}{-5} & \text{Divide both sides by -5.} \\ x & = & 7 & \text{Simplify} \end{array}$$

The solution set is $\{7\}$.

Two other important properties are:

Property 5: Symmetric Property

$$\text{If } A = B \text{ then } B = A.$$

Property 6: Transitive Property

$$\text{If } A = B \text{ and } B = C \text{ then } A = C.$$

When solving, we often see $2 = x$ but that is equivalent to $x = 2$.

4.2 Isolating the Variable

The idea behind solving in algebra is to isolate the variable. If given a linear equation of the form $ax + b = c$ then we can solve it in two steps. First use the equality property of addition or subtraction to isolate the variable term. Next isolate the variable by using the equality property of multiplication or division. The property choice depends on the given operation, we choose to apply the opposite property of the given operation. For example, if given a term plus three we would first choose to subtract three on both sides of the equation. If given two times the variable then we would choose to divide both sides by two.

Example 6
Solve $2x + 3 = 13$.

$$\begin{array}{rcll} 2x + 3 & = & 13 & \\ 2x + 3 - 3 & = & 13 - 3 & \text{Subtract 3 on both sides.} \\ 2x & = & 10 & \\ \frac{2x}{2} & = & \frac{10}{2} & \text{Divide both sides by 2.} \\ x & = & 5 & \end{array}$$

The solution set is $\{5\}$.

Example 7
Solve $-3x - 2 = 9$.

$$\begin{array}{rcll} -3x - 2 & = & 9 & \\ -3x - 2 + 2 & = & 9 + 2 & \text{Add 2 to both sides.} \\ -3x & = & 11 & \\ \frac{-3x}{-3} & = & \frac{11}{-3} & \text{Divide both sides by -3.} \\ x & = & -\frac{11}{3} & \end{array}$$

The solution set is $\{-\frac{11}{3}\}$

Example 8
Solve $\frac{x}{3} + \frac{1}{2} = \frac{2}{3}$.

$$\begin{array}{rcll} \frac{x}{3} + \frac{1}{2} & = & \frac{2}{3} & \\ \frac{x}{3} + \frac{1}{2} - \frac{1}{2} & = & \frac{2}{3} - \frac{1}{2} & \text{Subtract } \frac{1}{2} \text{ on both sides.} \\ \frac{x}{3} & = & \frac{2}{3}\left(\frac{2}{2}\right) - \frac{1}{2}\left(\frac{3}{3}\right) & \\ \frac{x}{3} & = & \frac{4}{6} - \frac{3}{6} & \\ \frac{x}{3} & = & \frac{1}{6} & \\ 3 \cdot \frac{x}{3} & = & 3 \cdot \frac{1}{6} & \text{Multiply both sides by 3.} \\ x & = & \frac{1}{2} & \end{array}$$

The solution set is $\{\frac{1}{2}\}$.

In order to retain the equality, we must perform the same operation on both sides of the equation. To isolate the variable we want to remember to choose the opposite operation not the opposite number. For example, if we have -5x = 20 then we choose to divide both sides by -5, not 5.

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Figure 1: Video Example 01

4.2.1 Multiplying by the Reciprocal

Recall that when multiplying reciprocals the result is 1, for example, $\left(\frac{3}{5}\right)\left(\frac{5}{3}\right) = \frac{15}{15} = 1$. We can use this fact when the coefficient of the variable is a fraction.

Example 9
Solve $-\frac{4}{5}x - 5 = 15$.

$$\begin{array}{ccll} -\frac{4}{5}x - 5 & = & 15 & \\ -\frac{4}{5}x - 5 + 5 & = & 15 + 5 & \text{Add 5 on both sides.} \\ -\frac{4}{5}x & = & 20 & \\ -\frac{5}{4} \cdot \left(-\frac{4}{5}x\right) & = & -\frac{5}{4} \cdot (20) & \text{Multiply both sides by } -\frac{5}{4}. \\ 1x & = & -5 \cdot 5 & \text{Simplify.} \\ x & = & -25 & \end{array}$$

The solution set is $\{-25\}$.

5 Combining Like Terms and Simplifying

Linear equations typically will not be given in standard form and thus will require some additional preliminary steps. These additional steps are to first simplify the expressions on each side of the equal sign using the order of operations.

5.1 Opposite Side Like Terms

Given a linear equation in the form $ax + b = cx + d$ we must combine like terms on opposite sides of the equal sign. To do this we will use the addition or subtraction property of equality to combine like terms on either side of the equation.

Example 10
Solve for y: $-2y+3=5y+17$

$$\begin{array}{rcll} -2y+3 & = & 5y+17 & \\ -2y+3-5y & = & 5y+17-5y & \text{Subtract 5y on both sides.} \\ -7y+3 & = & 17 & \\ -7y+3-3 & = & 17-3 & \text{Subtract 3 on both sides.} \\ -7y & = & 14 & \\ \frac{-7y}{-7} & = & \frac{14}{-7} & \text{Divide both sides by -7.} \\ y & = & -2 & \end{array}$$

The solution set is $\{-2\}$.

5.2 Same Side Like Terms

We will often encounter linear equations where the expressions on each side of the equal sign could be simplified. If this is the case then it is usually best to simplify each side first. After which we then use the properties of equality to combine opposite side like terms.

Example 11
Solve for a: $-4a+2-a=3+5a-2$

$$\begin{array}{rcll} -4a+2-a & = & 3+5a-2 & \text{Add same side like terms first.} \\ -5a+2 & = & 5a+1 & \\ -5a+2-5a & = & 5a+1-5a & \text{Subtract 5a from both sides.} \\ -10a+2 & = & 1 & \\ -10a+2-2 & = & 1-2 & \text{Subtract 2 from both sides.} \\ -10a & = & -1 & \\ \frac{-10a}{-10} & = & \frac{-1}{-10} & \text{Divide both sides by -10.} \\ a & = & \frac{1}{10} & \end{array}$$

The solution set is $\{\frac{1}{10}\}$

5.3 Simplifying Expressions First

When solving linear equations the goal is to determine what value, if any, will solve the equation. A general guideline is to use the order of operations to simplify the expressions on both sides first.

Example 12

Solve for x: $5(3x+2)-2=-2(1-7x)$

$$\begin{array}{rcll} 5(3x+2)-2 & = & -2(1-7x) & \text{Distribute.} \\ 15x+10-2 & = & -2+14x & \text{Add same side like terms.} \\ 15x+8 & = & -2+14x & \\ 15x+8-14x & = & -2+14x-14x & \text{Subtract 14x on both sides.} \\ x+8 & = & -2 & \\ x+8-8 & = & -2-8 & \text{Subtract 8 on both sides.} \\ x & = & -10 & \end{array}$$

The solution set is $\{-10\}$.

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Figure 2: Video Example 02

6 Conditional Equations, Identities, and Contradictions

There are three different kinds of equations defined as follows.

Definition 3: Conditional Equation
A conditional equation is true for particular values of the variable.

Definition 4: Identity
An identity is an equation that is true for all possible values of the variable. For example, x = x has a solution set consisting of all real numbers, $\Re$.

Definition 5: Contradiction
A contradiction is an equation that is never true and thus has no solutions. For example, x + 1 = x has no solution. No solution can be expressed as the empty set $\{\ \}=\varnothing$.

So far we have seen only conditional linear equations which had one value in the solution set. If when solving an equation and the end result is an identity, like say $0=0$, then any value will solve the equation. If when solving an equation the end result is a contradiction, like say $0=1$, then there is no solution.

Example 13

Solve for x: $4(x+5)+6=2(2x+3)$

$$\begin{array}{rcll} 4(x+5)+6 & = & 2(2x+3) & \text{Distribute} \\ 4x+20+6 & = & 4x+6 & \text{Add same side like terms.} \\ 4x+26 & = & 4x+6 & \\ 4x+26-4x & = & 4x+6-4x & \text{Subtract 4x on both sides.} \\ 26 & = & 6 & \text{False} \end{array}$$

There is no solution, $\varnothing$.

Example 14
Solve for y: $3(3y+5)+5=10(y+2)-y$

$$\begin{array}{rcll} 3(3y+5)+5 & = & 10(y+2)-y & \text{Distribute} \\ 9y+15+5 & = & 10y+20-y & \text{Add same side like terms.} \\ 9y+20 & = & 9y+20 & \\ 9y+20-20 & = & 9y+20-20 & \text{Subtract 20 on both sides.} \\ 9y & = & 9y & \\ 9y-9y & = & 9y-9y & \text{Subtract 9y on both sides.} \\ 0 & = & 0 & \text{True} \end{array}$$

The equation is an identity, the solution set consists of all real numbers, $\Re$.

7 Linear Literal Equations

Literal equations, or formulas, usually have more than one variable. Since the letters are placeholders for values, the steps for solving them are the same. Use the properties of equality to isolate the indicated variable.

Example 15
Solve for a: $P=2a+b$

$$\begin{array}{rcll} P & = & 2a+b & \\ P-b & = & 2a+b-b & \text{Subtract b on both sides.} \\ P-b & = & 2a & \\ \frac{P-b}{2} & = & \frac{2a}{2} & \text{Divide both sides by 2.} \\ \frac{P-b}{2} & = & a & \end{array}$$

Solution: $a=\frac{P-b}{2}$

Example 16
Solve for x: $z=\frac{x+y}{2}$

$$\begin{array}{rcll} z & = & \frac{x+y}{2} & \\ 2\cdot z & = & 2\cdot\frac{x+y}{2} & \text{Multiply both sides by 2.} \\ 2z & = & x+y & \\ 2z-y & = & x+y-y & \text{Subtract y on both sides.} \\ 2z-y & = & x & \end{array}$$

Solution $x=2z-y$

8 Exercises

8.1 Checking Solutions

Exercise 1 *(Solution on p. 12.)*

Is $x = 7$ a solution to $-3x + 5 = -16$?

Exercise 2 *(Solution on p. 12.)*

Is $x = 2$ a solution to $-2x - 7 = 28$?

Exercise 3 *(Solution on p. 12.)*

Is $x = -3$ a solution to $\frac{1}{3}x - 4 = -5$?

Exercise 4 *(Solution on p. 12.)*

Is $x = -2$ a solution to $3x - 5 = -2x - 15$?

Exercise 5 *(Solution on p. 12.)*

Is $x = -\frac{1}{2}$ a solution to $3(2x + 1) = -4x - 3$?

8.2 Solving in One Step

Exercise 6 *(Solution on p. 12.)*

Solve for x: $x - 5 = -8$

Exercise 7 *(Solution on p. 12.)*

Solve for y: $-4 + y = -9$

Exercise 8 *(Solution on p. 12.)*

Solve for x: $x - \frac{1}{2} = \frac{1}{3}$

Exercise 9 *(Solution on p. 12.)*

Solve for x: $x + 2\frac{1}{2} = 3\frac{1}{3}$

Exercise 10 *(Solution on p. 12.)*

Solve for x: $4x = -44$

Exercise 11 *(Solution on p. 12.)*

Solve for a: $-3a = -30$

Exercise 12 *(Solution on p. 12.)*

Solve for y: $27 = 9y$

Exercise 13 *(Solution on p. 12.)*

Solve for x: $\frac{x}{3} = -\frac{1}{2}$

Exercise 14 *(Solution on p. 12.)*

Solve for t: $-\frac{t}{12} = \frac{1}{4}$

Exercise 15 *(Solution on p. 12.)*

Solve for x: $\frac{7}{3}x = -\frac{1}{2}$

8.3 Solve in Two Steps

Exercise 16 *(Solution on p. 12.)*
Solve for a: $3a - 7 = 23$

Exercise 17 *(Solution on p. 12.)*
Solve for y: $-3y + 2 = -13$

Exercise 18 *(Solution on p. 12.)*
Solve for x: $-5x + 8 = 8$

Exercise 19 *(Solution on p. 12.)*
Solve for x: $\frac{1}{2}x + \frac{1}{3} = \frac{2}{5}$

Exercise 20 *(Solution on p. 12.)*
Solve for y: $3 - 2y = -11$

Exercise 21 *(Solution on p. 13.)*
Solve for x: $-10 = 2x - 5$

Exercise 22 *(Solution on p. 13.)*
Solve for a: $4a - \frac{2}{3} = -\frac{1}{6}$

Exercise 23 *(Solution on p. 13.)*
Solve for x: $\frac{3}{5}x - \frac{1}{2} = \frac{1}{10}$

Exercise 24 *(Solution on p. 13.)*
Solve for y: $-\frac{4}{5}y + \frac{1}{3} = \frac{1}{15}$

Exercise 25 *(Solution on p. 13.)*
Solve for x: $-x - 5 = -2$

8.4 Solve in Multiple Steps

Exercise 26 *(Solution on p. 13.)*
Solve for x: $3x - 5 = 2x - 17$

Exercise 27 *(Solution on p. 13.)*
Solve for y: $-2y - 7 = 3y + 13$

Exercise 28 *(Solution on p. 13.)*
Solve for a: $\frac{1}{2}a - \frac{2}{3} = a + \frac{1}{5}$

Exercise 29 *(Solution on p. 13.)*
Solve for x: $-2 + 4x + 9 = 7x + 8 - 2x$

Exercise 30 *(Solution on p. 13.)*
Solve for a: $3a + 5 - x = 2a + 7$

Exercise 31 *(Solution on p. 13.)*
Solve for b: $-7b + 3 = 2 - 5b + 1 - 2b$

Exercise 32 *(Solution on p. 13.)*
Solve for y: $-5(2y - 3) + 2 = 12$

Exercise 33 *(Solution on p. 13.)*
Solve for x: $3 - 2(x + 4) = -3(4x - 5)$

Exercise 34 *(Solution on p. 13.)*
Solve for a: $-3(2a - 3) + 2 = 3(a + 7)$

Exercise 35 *(Solution on p. 13.)*
Solve for x: $10(3x + 5) - 5(4x + 2) = 2(5x + 20)$

8.5 Literal Equations

Exercise 36 *(Solution on p. 13.)*

Solve for w: $P = 2l + 2w$

Exercise 37 *(Solution on p. 13.)*

Solve for b: $P = a + b + c$

Exercise 38 *(Solution on p. 13.)*

Solve for C: $F = \frac{9}{5}C + 32$

Exercise 39 *(Solution on p. 13.)*

Solve for r: $C = 2\pi r$

Exercise 40 *(Solution on p. 13.)*

Solve for y: $z = \frac{x-y}{5}$

Solutions to Exercises in this Module

Solution to Exercise 1 (p. 9)
Yes

Solution to Exercise 2 (p. 9)
No

Solution to Exercise 3 (p. 9)
Yes

Solution to Exercise 4 (p. 9)
Yes

Solution to Exercise 5 (p. 9)
No

Solution to Exercise 6 (p. 9)
$x = -3$

Solution to Exercise 7 (p. 9)
$y = -5$

Solution to Exercise 8 (p. 9)
$x = \frac{5}{6}$

Solution to Exercise 9 (p. 9)
$x = \frac{5}{6}$

Solution to Exercise 10 (p. 9)
$x = -11$

Solution to Exercise 11 (p. 9)
$a = 10$

Solution to Exercise 12 (p. 9)
$y = 3$

Solution to Exercise 13 (p. 9)
$x = -\frac{3}{2}$

Solution to Exercise 14 (p. 9)
$t = -3$

Solution to Exercise 15 (p. 9)
$x = -\frac{3}{14}$

Solution to Exercise 16 (p. 10)
$a = 10$

Solution to Exercise 17 (p. 10)
$y = 5$

Solution to Exercise 18 (p. 10)
$x = 0$

Solution to Exercise 19 (p. 10)
$x = \frac{2}{15}$

Solution to Exercise 20 (p. 10)
$y = 7$

Solution to Exercise 21 (p. 10)
$x = -\frac{5}{2}$

Solution to Exercise 22 (p. 10)
$a = \frac{1}{8}$

Solution to Exercise 23 (p. 10)
$x = 1$

Solution to Exercise 24 (p. 10)
$y = \frac{1}{3}$

Solution to Exercise 25 (p. 10)
$x = -3$

Solution to Exercise 26 (p. 10)
$x = -12$

Solution to Exercise 27 (p. 10)
$y = -4$

Solution to Exercise 28 (p. 10)
$a = -\frac{26}{15}$

Solution to Exercise 29 (p. 10)
$x = -1$

Solution to Exercise 30 (p. 10)
No Solution, $\varnothing$

Solution to Exercise 31 (p. 10)
All Reals, $\Re$

Solution to Exercise 32 (p. 10)
$y = \frac{1}{2}$

Solution to Exercise 33 (p. 10)
$x = 2$

Solution to Exercise 34 (p. 10)
$a = -\frac{10}{9}$

Solution to Exercise 35 (p. 10)
All Reals, $\Re$

Solution to Exercise 36 (p. 11)
$w = \frac{P-2l}{2}$

Solution to Exercise 37 (p. 11)
$b = P - a - c$

Solution to Exercise 38 (p. 11)
$C = \frac{5F-160}{9}$

Solution to Exercise 39 (p. 11)
$r = \frac{C}{2\pi}$

Solution to Exercise 40 (p. 11)
$y = -5z + x$